ELSEVIER
Journal of Geometry and Physics 30 (1999) 187–203
JOURNAL OF GEOMETRY AND PHYSICS

On the geometry of Lagrangian mechanics with non-holonomic constraints

Joseph Grifone [a,*], Mohamad Mehdi [b,1]

[a] *Laboratoire E. Picard, UMR CNRS 5580, Département de Mathématiques, Université Paul Sabatier 118, route de Narbonne, 31062 Toulouse Cedex, France*

[b] *Université Libanaise, Faculté des Sciences, Choran, BP 135292 Beyrouth, Liban*

Received 17 October 1997

In memory of Professor Joseph Klein

Abstract

As is well known, Lagrangian mechanics have been entirely geometrized in terms of symplectic geometry. On the other hand, the geometrization of non-holonomic mechanics has been less developed. However, due to the interest aroused by non-holonomic geometry, many papers have been devoted to this subject. In this article we generalize the construction of a connection whose geodesics are the trajectories of a system, obtained by Vershik and Feddeef in the case where the Lagrangian is quadratic and the constraints are linear on the velocities. Using the algebraic formalism of the connections theory introduced by the first author, we carry out the construction in the general case of an arbitrary mechanical system (i.e. of a manifold with a convex Lagrangian not necessarily homogeneous) with ideal non-holonomic constraints. Moreover, we prove something stronger than the result of Vershik and Feddeev: our connection has not only the above-mentioned property for the geodesics, but it preserves the Hamiltonian by parallel transport. This connection is then a generalization of the Levi–Civita connection for the Riemannian manifolds for which the metric (i.e. the kinetic energy) is preserved by parallel transport. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Dynamical systems
1991 MSC: 58A10; 58F05; 53B05; 70H35.
Keywords: Lagrangian Mechanics; Non-holonomic constraints; Non-holonomic geometry; Connections; Sprays

* Corresponding author. E-mail: grifone@picard.ups-tlse.fr
[1] E-mail: lusc1@lara.cnrs.edu.lb

PII: S0393-0440(98)00055-2

1. A survey of the algebraic formalism for the theory of connections

In this article we follow the ideas of Gallissot [3] and Klein [4], for the geometrization of Lagrangian mechanics, in which the framework of the tangent bundle plays an essential role. In the last part of his thesis ([4], ch. 6), Klein applied his point of view to non-holonomic Lagrangian systems, obtaining, among other results, a geometric presentation of the principle of minimal curvature. Recently, owing to the interest aroused by non-holonomic geometry, many papers have been devoted to this subject (cf. for example [1,7,8,10–13], and [9], which contains a large bibliography).

Our aim is to set up a connection whose geodesics are the trajectories of the system and which preserves the Hamiltonian by parallel transport. Moreover, we suppose that the Lagrangian is an arbitrary convex function without homogeneity hypothesis. As we will see, even in the simplest case when the Lagrangian is the sum of a quadratic form and a potential which does not depend on the velocities, the torsion of the connection does not vanish. Therefore the non-vanishing of the torsion is essentially related to the deviation from the homogeneous case.

In this paper we use the algebraic formalism for the theory of connections introduced in [6], which is based on the Frölicher–Nijenhuis theory of derivations associated with vector-valued forms (cf. [2]). We recall in this section the essential notions of this formalism.

1.1. Connections

Let M be a differentiable manifold. A p-form $\omega \in \otimes^p T^*TM$ is called *semi-basic* if $\omega(X_1, \ldots, X_p) = 0$ whenever one of the vectors X_i is vertical. In the same way, a vector-valued l-form L (i.e. $L \in \otimes^l T^*TM \otimes TTM$) is called *semi-basic* if it is vertical valued and $L(X_1, \ldots, X_l) = 0$ whenever one of the vectors X_i is vertical. We shall denote by T^v the vertical space, T_v^* the space of the semi-basic 1-forms and $\Lambda^p T_v^*$ the space of the skew-symmetric semi-basic p-forms. In local adapted coordinates (x^α, y^α) [2] scalar and vector semi-basic forms have the expression, respectively:

$$\omega = \omega_{i_1 \cdots i_p}(x, y)\, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_p},$$

$$L = L^j_{i_1, \ldots, i_l}(x, y)\, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_l} \otimes \frac{\partial}{\partial y^j}.$$

Let $\pi : TM \longrightarrow M$ be the tangent bundle to M and $P : TTM \longrightarrow TM$ the tangent bundle to TM. We have the exact sequence:

$$0 \longrightarrow TM \underset{M}{\times} TM \stackrel{i}{\longrightarrow} TTM \stackrel{j}{\longrightarrow} TM \underset{M}{\times} TM \longrightarrow 0,$$

[2] The y^α are the co-ordinates of a vector of TM on the basis $(\partial/\partial x^\alpha)$.

where $i(v, w) := \frac{\mathrm{d}}{\mathrm{d}t}(v + tw)\big|_{t=0}$ is the natural injection in the vertical bundle T^v and $j := (P, \pi)$. The (1–1) tensor $J := i \circ j$ on TM is called *almost tangent structure* or *vertical endomorphism*. In adapted local coordinates

$$J = \mathrm{d}x^\alpha \otimes \frac{\partial}{\partial y^\alpha}.$$

The following identities hold:

$$J^2 = 0 \quad \text{and} \quad [J, J] = 0.$$

Note that Ker J = Im $J = T^v$.

The vertical field $C_z := i(z, z)$ is called *canonical field* or *Liouville field*. In local coordinates

$$C = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

Definition 1.1. A spray is a vector field S on TM such that $JS = C$.

Locally:

$$S = \frac{\partial}{\partial x^\alpha} + f^\alpha(x, y)\frac{\partial}{\partial y^\alpha}.$$

With any spray a system of ordinary second-order differential equations is associated, and conversely a spray is associated with any system of ordinary second-order differential equations, in the following way. A curve $\gamma : I \longrightarrow M$ such that γ' is an integral curve of S, i.e. $S_{\gamma'} = \gamma''$, is called a *path* of S. Locally if $\gamma : t \longmapsto x^\alpha(t)$ is a path of S, then the x^α verify the second-order equations:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} = f^\alpha\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right).$$

Conversely, if we have a system of ordinary second-order equations, the spray is defined by this formula and this definition does not depend on the local coordinates. [3]

Let L be a (scalar or vector-valued) semi-basic l-form, $l \geq 1$; the *potential* L^0 of L is the $(l-1)$ form defined by $L^0 = i_S L$ where S is an arbitrary spray. Obviously, L^0 does not depend on the choice of S.

A spray S is called homogeneous if $[C, S] = S$. This condition means that the functions $f^\alpha(x, y)$ are homegeneous of degree 2 in the variables y^α. If, in addition, S is C^2 on the zero section, the f^α are quadratic in the y^α. The vertical vector field $S^* := [C, S] - S$ which estimates the non-homogeneity of S is called *deflection*.

Definition 1.2. A connection on M is a (1–1) tensor field Γ on TM such that

$$J\Gamma = J \quad \text{and} \quad \Gamma J = -J.$$

[3] The sprays are also called SODE (second-order differential equations), cf. for example [11]. In [6] they are called *semi-sprays*, and the term *spray* is reserved to homogeneous second-order equations.

The connection is called linear if $[C, \Gamma] = 0$ (if Γ is C^∞ on $TM \setminus \{0\}$ but is not C^1 on the zero section, the connection is called homogeneous).

In the basis $(\partial/\partial x^\alpha, \partial/\partial y^\alpha)$, a connection is represented by the matrix

$$\begin{pmatrix} \delta^\alpha_\beta & 0 \\ -2\Gamma^\alpha_\beta & -\delta^\alpha_\beta \end{pmatrix}.$$

The fuctions Γ^α_β are called the *coefficients of the connection.* If the connection is linear, then the Γ^α_β are linear in the y^γ: $\Gamma^\alpha_\beta(x, y) = y^\gamma \Gamma^\beta_{\alpha\gamma}(x)$. If the connection is homogeneous, the Γ^α_β are homogeneous of degree 1 in the y^γ.

The semi-basic tensor $H = (1/2)[C, \Gamma]$ which estimates the non-homogeneity of the connection is called *tension.*

It is easily verified that $\Gamma^2 = I$ and that the eigenspace corresponding to the eigenvalue -1 is the vertical space. Then TTM splits into the direct sum

$$TTM = T^h \oplus T^v,$$

where T^h is the eigenspace corresponding to the eigenvalue $+1$, called *horizontal space.* We denote by h and v the horizontal and vertical projectors:

$$h := \tfrac{1}{2}(I + \Gamma), \qquad v := \tfrac{1}{2}(I - \Gamma).$$

Consider two manifolds N and M, and two C^∞ maps $\gamma : N \to M$ and $z : N \to TM$ such that $\pi \circ z = \Gamma$:

$$\begin{array}{ccc} & & TM \\ & {\scriptstyle z}\nearrow & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\gamma} & M. \end{array}$$

If $w \in \mathfrak{X}(N)$ and $\xi_z : T^v_z \to T_{\pi z}M$ is the natural isomorphism, the *covariant derivative* of z with respect to w is defined by

$$D_{w(x)}z = \xi_{z(x)}(v \circ z_* \circ w).$$

In particular, for $N = M$, $\gamma = id$ and w, z two vector fields on M, we have

$$D_w z = w^\alpha \left(\frac{\partial z^\beta}{\partial x^\alpha} + \Gamma^\beta_\alpha(x, z(x))\right)\frac{\partial}{\partial x^\beta}.$$

Let $N = I$ be an interval of $\mathbb{R}$, $w = \mathrm{d}/\mathrm{d}t$, $\gamma : I \to M$ a curve on M and $z : I \to TM$ a vector field along γ. We have

$$D_{\mathrm{d}/\mathrm{d}t}z = \left(\frac{\mathrm{d}z^\beta}{\mathrm{d}t} + \Gamma^\beta_\alpha(\gamma(t), z(t))\frac{\mathrm{d}\gamma^\alpha}{\mathrm{d}t}\right)\frac{\partial}{\partial x^\beta}.$$

A vector field $z \in TM$ along a curve γ is called *parallel* if $D_{\mathrm{d}/\mathrm{d}t}z = 0$, i.e. $vz' = 0$. A *geodesic* is a curve γ such that

$$D_{\mathrm{d}/\mathrm{d}t}\gamma' = 0,$$

which means that the acceleration is horizontal:

$$v\gamma'' = 0.$$

A spray S is canonically associated with a connection. It is defined by

$$S := h\tilde{S},$$

where $\tilde{S}$ is an arbitrary spray (S does not depend on the choice of $\tilde{S}$). It is easy to check that *the paths of the spray associated with a connection are the geodesics of the connection.*

Proposition 1.3 (cf. [6]). *If S is a spray then $\Gamma := [J, S]$ is a connection.*

It is important to note that this connection is not, in general, the most appropriate connection associated with S. Indeed, the spray of $[J, S]$ is not S, but $(1/2)(S + [C, S])$. Then *the geodesics of* $[J, S]$ *are not the paths* S, unless S is homogeneous. In other words, if a spray S – i.e. a system of ordinary second-order equations – is given, the geodesics of the connection $[J, S]$ are not solutions of this system, unless the equations are homogeneous. The notion of torsion is introduced in order to construct a connection whose geodesics are the solutions of a second-order non-homogeneous system (cf. [6]).

Definition 1.4. The semi-basic vector-valued 2-form $t := (1/2)[J, \Gamma]$ is called the weak torsion of Γ. The strong torsion is the semi-basic vector-valued 1-form $T := t^{\mathrm{o}} - H$, where H is the tension of the connection.

It is easy to see that the strong torsion "equilibrates" the spray S of the connection, in the sense that its potential compensates the deflection of S, i.e.

$$T^0 + S^* = 0.$$

The following result holds:

Theorem 1.5 [6, (I.55)]. *Let S be a spray and T a semi-basic* (1–1) *tensor field which equilibrates S. Then there exists one and only one connection Γ whose spray is S and whose strong torsion is T. It is given by*

$$\Gamma = [J, S] + T.$$

Note that $T = 0$ implies $\Gamma = [J, S]$, and then $[J, \Gamma] = [J, [J, S]] = (1/2)[[J, J], S] = 0$; so $t = 0$, and consequently, $H = 0$. Conversely, if H and t vanish, $T = 0$, so $T = 0$ if and only if $t = 0$ and the connection is homogeneous. When the connection is linear, T and t agree with the usual torsion, up to the identification of the semi-basic tensors at a point $z \in TM$ with the tensors on $T_{\pi z}M$.

Then:

between "S" and "arc" the connections whose spray is S and whose geodesics are the solutions of a given second order system are given by the formula $\Gamma = [J, S] + T$ where T is a semi-basic (1–1) *tensor field TM, equilibrating S.*

Every connection Γ on M determines an almost-complex structure F on TM which permutes the vertical and horizontal spaces; F is the unique (1–1) tensor field on TM such that

$$FJ = h \quad \text{and} \quad Fh = -J.$$

F and the spray of the connection are related by $F = h[S, h] - J$.

Finally, the *curvature* is the semi-basic vector-valued 2-form $R := -(1/2)[h, h]$. For a linear connection this agrees with the usual curvature, up to the identification of the semi-basic tensor at $z \in TM$ with the tensors at $T_{\pi z}M$.

1.2. Lagrangians

Definition 1.6. A Lagrangian is a map $E : TM \to R$ which is C^∞ on $TM \setminus \{0\}$. E is called regular if the 2-form $\Omega := \mathrm{d}\mathrm{d}_J E$ is symplectic.

Locally, E is regular if and only if

$$\det \left\| \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} \right\| \neq 0.$$

For example, if g is a Riemannian (or pseudo-Riemannian) metric on M, the quadratic form $E(v) := (1/2)g(v, v)$ on TM is a regular Lagrangian. If E is homogeneous of degree 2, i.e. $\mathcal{L}_C E = 2E$, and E is C^1 on the zero section, E defines a Finsler structure. [4]

Proposition 1.7 (cf. [5; 6, (II.15)]). *Let E be a regular Lagrangian and $\mathcal{H} := \mathcal{L}_C E - E$ be the associated* Hamiltonian. *The vector field S on TM defined by*

$$i_S \Omega = -\mathrm{d}\mathcal{H}$$

is a spray, called canonical spray. *The connection $\Gamma = [J, S]$ is called the* natural connection *associated with E. In particular, if E is the quadratric form associated with a Riemannian (or pseudo-Riemannian) metric, we get the Levi-Civita connection.*

Locally, the components f^α of the canonical spray E are

$$f^\alpha = g^{\alpha\beta} \left(\frac{\partial E}{\partial x^\beta} - y^\lambda \frac{\partial^2 E}{\partial x^\lambda \partial y^\beta} \right),$$

where the $g^{\alpha\beta}$ are the coefficients of the inverse of the matrix $\| \partial^2 E / \partial y^\alpha \partial y^\beta \|$. An easy computation shows that the paths of S are the solutions of the Euler–Lagrange system defined by E:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial E}{\partial \dot{x}^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0.$$

[4] If E is C^2 and homogeneos of degreee 2, it is quadratic and then it defines a Riemannian (pseudo-Riemannian) structure.

Note that if E is homogeneous, then S is homogeneous too and the paths of S are the geodesics of Γ. Inversely, if E is not homogeneous, the paths of S are not the geodesics of the connection. In other words:

If E is not homogeneous, the geodesics of $[J, S]$ are not the solutions of the Euler–Lagrange equations.

It is well known that a regular Lagrangian E allows to define a metric (pseudo-Riemannian in general) on the vertical bundle, by setting

$$g(JX, JY) = \Omega(JX, Y) \quad \text{for } X, Y \in TTM,$$

where $\Omega = \mathrm{d}\mathrm{d}_J E$. If Γ is a connection on M, g can be extended to TM by putting

$$g_\Gamma(X, Y) = g(JX, JY) + g(vX, vY),$$

where v is the vertical projector (cf. [6]). It easy to see that the horizontal and vertical spaces are orthogonal with respect to g_Γ. This is the reason why g_Γ is called the *adapted metric to Γ*.[5] If E is convex, i.e. the matrix $\| \partial^2 E/\partial y^\alpha \partial y^\beta \|$ is positive-definite, then g_Γ is a Riemannian metric.

Finally, we have

$$g_\Gamma(X, Y) = \Omega(X, FY),$$

where F is the almost-complex structure associated with Γ.

2. Lagrangian connections

As we said, if E is a non-homogeneous Lagrangian, the geodesics of the "natural" connection $[J, S]$ are not the paths of S, i.e. the solutions of the Euler–Lagrange equations. The notion of *Lagrangian connection* allows us to remove this difficulty.

Definition 2.1. A connection is called Lagrangian if the horizontal space is Lagrangian with respect to the symplectic form $\Omega = \mathrm{d}\mathrm{d}_J E$, i.e. if $\Omega(hX, hY) = 0$ for any $X, Y \in TTM$.

One can easily check that Γ is Lagrangian if and only if $i_\Gamma \Omega = 0$, which is equivalent to $i_h \Omega = \Omega$, to $i_v \Omega = \Omega$ and to $i_F \Omega = 0$.

Remark. *If E is homogeneous of degree 2, the natural connection $[J, S]$ is Lagrangian.*

Indeed

$$i_{[J,S]}\Omega = i_S \mathrm{d}_J \Omega + \mathrm{d}_J i_S \Omega - \mathcal{L}_C \Omega.$$

[5] This metric is frequently called Sasaki metric.

Now $\mathrm{d}_J\Omega = 0$ and $\mathcal{L}_C\Omega = \Omega$, because E is homogeneous of degree 2; thus

$$i_{[J,S]}\Omega = \mathrm{d}_J\mathrm{d}E + \Omega = 0.$$

The interest of the Lagrangian connection comes into sight from the following property:

Proposition 2.2 (cf. [6, I, II.32, II.33]). *Consider a regular Lagrangian E, $\mathcal{H} = \mathcal{L}_C E - E$ the associated Hamiltonian and Γ a Lagrangian connection. Then the spray of Γ is the canonical spray if and only if*

$$\mathrm{d}_h\mathcal{H} = 0.$$

Indeed, let S be the spray of Γ ; we have

$$i_h(i_S\Omega + \mathrm{d}\mathcal{H}) = i_h i_s\Omega + \mathrm{d}_h\mathcal{H} = i_s i_h\Omega - i_{hS}\Omega + \mathrm{d}_h\mathcal{H}.$$

Now $i_h\Omega = \Omega$, beacuse Γ is Lagrangian and $hS = S$, S being the spray of Γ; then

$$i_h(i_S\Omega + \mathrm{d}\mathcal{H}) = \mathrm{d}_h\mathcal{H}.$$

On the other hand, it is easy to see that the 1-form $\omega = i_S\Omega + \mathrm{d}\mathcal{H}$ is semi-basic and hence $i_h\omega = \omega$. Then we have

$$i_S\Omega + \mathrm{d}\mathcal{H} = \mathrm{d}_h\mathcal{H},$$

which proves that S is the canonical spray if and only if $\mathrm{d}_h\mathcal{H} = 0$.

Interpretation. This property means that the Hamiltonian is preserved by parallel transport. Indeed, let us consider the set of the vector fields along a curve $\gamma : I \to M$, that is the set of the $z : I \to TM$ such that $\pi \circ z(t) = \gamma(t)$. The following corollary holds:

Corollary 2.3. *Let S be the canonical spray and Γ be a Lagrangian connection whose spray is S. Consider a curve $\gamma : I \to M$. For every parallel vector field z along γ, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) = 0.$$

In other words, the Hamiltonian is preserved by parallel transport.

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) = \mathrm{d}\mathcal{H}(z'(t)) = \mathrm{d}\mathcal{H}(hz'(t))$$

because z is parallel. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) = (\mathrm{d}_h\mathcal{H})(z'(t)) = 0.$$

Remark. *As it appears from the above remark, in the Riemannian case the natural connection – which actually is the Levi-Civita connection – is Lagrangian, hence the Hamiltonian* *is preserved by parallel transport. Since* $\mathcal{H}(z) = E(z) = (1/2)g(z, z)$, *the property* $(\mathrm{d}/\mathrm{d}t)\mathcal{H}(z(t)) = 0$ *is equivalent to*

$$Dg = 0,$$

which is the essential property of the Levi-Civita connection. Then the Lagrangian connections whose spray is the canonical one, generalize, to the non-homogeneous Lagrangians, the property $Dg = 0$.

The following result gives a generalization of the construction of the Levi-Civita connection for general Lagrangians, even non-homogeneous.

Theorem 2.4. *Let E be a regular Lagrangian and $\tilde{S}$ a spray. Then there exists a Lagrangian connection whose spray is $\tilde{S}$. In particular (taking for $\tilde{S}$ the canonical spray), there exists a connection whose geodesics satify the Euler–Lagrange equation and such that the Hamiltonian is preserved by parallel transport.*

Indeed, let $\tilde{\Gamma} = [J, \tilde{S}] + T$ be a connection whose spray is $\tilde{S}$. Then $\tilde{\Gamma}$ is Lagrangian if and only if $i_{\tilde{\Gamma}}\Omega = 0$, i.e. $i_{[J,\tilde{S}]}\Omega + i_T\Omega = 0$. Let S be the canonical spray and put $U = S - \tilde{S}$. We get $i_{[J,\tilde{S}]}\Omega + i_T\Omega = i_{[J,S]}\Omega - i_{[J,U]}\Omega + i_T\Omega$. On the other hand, since S is the canonical spray, we obtain

$$i_{[J,S]}\Omega = i_J L_S \Omega = i_J \mathrm{d} i_S \Omega = -i_J \mathrm{d}^2\mathcal{H} = 0.$$

Then the condition ensuring that $\tilde{\Gamma}$ is Lagrangian is

$$i_T\Omega - i_{[J,U]}\Omega = 0.$$

Since $i_{[J,U]}\Omega = i_J \mathcal{L}_U \Omega$, this condition can be written as

$$g(TX, JY) + \mathcal{L}_U\Omega(JX, Y) = g(TY, JX) + \mathcal{L}_U\Omega(JY, X), \tag{1}$$

because $\Omega(vX, Y) = g(vX, JY)$. If we put

$$\Theta(X, Y) = g(TX, JY) + \mathcal{L}_U\Omega(JX, Y), \tag{2}$$

condition (1) can be expressed by saying that the scalar semi-basic 2-form Θ is symmetric. Since T has to "equilibrate" $\tilde{S}$, the problem reduces to finding a scalar *semi-basic* and *symmetric* 2-form Θ such that

$$\Theta(S, Y) = -g(\tilde{S}^*, JY) + (\mathcal{L}_U\Omega)(C, Y). \tag{3}$$

As soon as Θ has been fixed, one defines, following (2), the strong torsion T by

$$g(TX, JY) = \Theta(X, Y) - (\mathcal{L}_U\Omega)(JX, Y).$$

Let us consider the scalar semi-basic 2-forms $\Theta = i_C\Omega \odot \omega$, where ω is a scalar semi-basic 1-form and $\odot$ is the symmetric product. Let us show that ω can be taken in such a way that (3) is satisfied. Now (3) is satisfied if and only if

$$\omega^o i_C\Omega + g(C, C)\omega = -i_{\tilde{S}^*}\Omega + i_C\mathcal{L}_U\Omega. \tag{4}$$

Then, taking the potential of both sides of this equation, we have

$$2\omega^o g(C, C) = -g(\tilde{S}^*, C) - (\mathcal{L}_U \Omega)(S, C).$$

Replacing it in (4), we find

$$\omega = \frac{1}{g(C, C)} \left[-i_{\tilde{S}^*} \Omega + i_C \mathcal{L}_U \Omega + \frac{g(\tilde{S}^*, C) + \mathcal{L}_U \Omega(S, C)}{2g(C, C)} i_C \Omega \right],$$

which determines Θ and, consequently, T.

For example, in the case where $\tilde{S}$ is the canonical spray (i.e. $U = 0$), we get

$$T = \frac{1}{(g(C, C))^2} [g(\tilde{S}^*, C) i_C \Omega \otimes C - g(C, C) i_{\tilde{S}^*} \Omega \otimes C - g(C, C) i_C \Omega \otimes \tilde{S}^*]. \tag{5}$$

3. Non-holonomic constrained systems

This section is, partially, a reformulation in our formalism of some definitions and results of [13].

Definition 3.1. An admissible non-holonomic constraint (or, simply, a non-holonomic constraint) is a submanifold $\mathcal{A}$ of TM everywhere transval to the vertical bundle, that is such that at any point of $\mathcal{A}$ one has

$$T\mathcal{A} + T^v = TTM,$$

where T^v is the vertical bundle.

The semi-basic 1-forms $\omega \in J^*(T\mathcal{A})^o$ are called reaction forces.

$\mathcal{A}$ is called ideal if the canonical field C is tangent to $\mathcal{A}$.

An admissible spray for the constraint $\mathcal{A}$ is a vector field $S' \in \mathfrak{X}(\mathcal{A})$ such that $JS' = C$.

Remarks.

1. *The transversality condition implies of course that* $\dim \mathcal{A} \geq n = \dim M$. *It expresses that* $\mathcal{A}$ *is fibered on* M *by the restriction to* $\mathcal{A}$ *of the projection of* TM *on* M.[6]
2. *The reaction forces are the 1-forms* $\omega \in T^*(TM)$ *such that* $\omega = i_J \alpha$ *with* $\alpha \in T^*TM$ *and* $\alpha(X) = 0$ *for any* $X \in T\mathcal{A}$. *If* $\mathcal{A}$ *is defined as the kernel of the submersion* $f = (f_1, \ldots, f_{n-k}) : TM \to R^{n-k}$, *where the* $f_i \in C^\infty(TM)$, *then* $(T\mathcal{A})^o$ *is the set of the 1-forms* α *on* T^*M *such that* $\alpha = \lambda_1 \mathrm{d} f_1 + \cdots + \lambda_{n-k} \mathrm{d} f_{n-k}$. *Then the reaction forces have the expression*

$$\omega = \lambda_1 \mathrm{d}_J f_1 + \cdots + \lambda_{n-k} \mathrm{d}_J f_{n-k}.$$

[6] Under the natural hypothesis that $\mathcal{A}$ is connected and smooth.

3. *The condition that the constraint is ideal comes from [13]. It expresses that the work of the reaction forces is 0. Indeed,* $C \in T\mathcal{A}$ *if and only if* $\alpha(C) = 0$ *for any* $\alpha \in T^*TM$ *such that* $\alpha(X) = 0\ \forall X \in T\mathcal{A}$, *and that is equivalent to* $\omega(S) = 0$ *for any reaction force* ω, *where S is an arbitrary spray. Now* $\omega(S) = 0$ *expresses in fact that the work of* ω *is 0, along all the curves.*

   If $\mathcal{A}$ *is defined by a submersion* $F = (f_1, \ldots, f_{n-k}) : TTM \to R^{n-k}$, *the condition that* $\mathcal{A}$ *is ideal can be expressed by* $\mathcal{L}_C F = 0 \bmod(F)$. *If* $f_1, \ldots, f_{n-k}$ *are homogeneous this condition is satified and hence* $\mathcal{A}$ *is ideal. In particular, linear constraints are ideal.*
4. *If* S' *is an admissible spray, its integral curves project on curves on M verifying a second-order equation constrained by the condition that the velocity is in* $\mathcal{A}$. *Conversely any second-order equation on M such that the velocities of the solutions are in* $\mathcal{A}$ *defines a spray which is admissible for* $\mathcal{A}$.

The transversality condition is the usual one on the non-holonomic constraints (cf. [9,13]). In fact we have:

Proposition 3.2. *Let* $(T\mathcal{A})^o$ *be the annihilator of* $T\mathcal{A}$. *The following properties are equivalent:*
(1) $\mathcal{A}$ *is an admissible non-holonomic constraint.*
(2) $(T\mathcal{A})^o$ *does not contain semi-basic 1-forms.*
(3) $\dim(T\mathcal{A})^o = \dim J^*(T\mathcal{A})^o$.
(4) $\dim T\mathcal{A} = n + \dim(T\mathcal{A} \cap T^v)$.

Indeed, taking the annihilator of the admissibility condition $T\mathcal{A} + T^v = TTM$, one gets $(T\mathcal{A})^o \cap (T^v)^o = \{0\}$ where $(T^v)^o$ is the annihilator of T^v, i.e. the set T_v^* of the semi-basic 1-forms, which proves that (1). is equivalent to (2).

On the other hand, let

$$\tilde{J} : (T\mathcal{A})^o \to J^*(T\mathcal{A})^o$$

be defined by $\tilde{J}\omega = J^*\omega$, where $(J^*\omega)(X) := \omega(JX)$. Property (3). expresses that $\tilde{J}$ is injective. Now $\operatorname{Ker} \tilde{J} = (T\mathcal{A})^o \cap (T^v)^o$. Then (3). is equivalent to (2).

Finally, if $T\mathcal{A} + T^v = TTM$ we have $\dim T\mathcal{A} + \dim T^v - \dim(T\mathcal{A} \cap T^v) = 2n$, hence $\dim T\mathcal{A} = n + \dim T\mathcal{A} \cap T^v$. Conversely, if this equality holds, there exist n independent vectors in $T_z\mathcal{A}$ which are not in T_z^v. Completing these n vectors by a basis of T_z^v we obtain a basis of T_zTM, which shows that $T_z\mathcal{A} + T_z^v = T_zTM$.

Let E be a convex Lagrangian, Γ a connection on M and g_Γ the (Riemannian) metric on TM adapted to the connection Γ (cf. Section 1).[7] Let us denote by $T^\perp\mathcal{A}$ the normal bundle on $\mathcal{A}$ defined by the orthogonality with respect to g_Γ. We have:

Proposition 3.3. *The following properties are equivalent:*
(1) $\mathcal{A}$ *is an admissible non-holonomic constraint.*

[7] In what follows, we shall take for Γ the Lagrangian connection with canonical spray, produced in Section 2. However, all that follows holds for an arbitrary connection.

(2) $T^{\perp}\mathcal{A} \cap T^{\mathrm{h}} = \{0\}$ *where* T^h *is the horizontal distribution.*
(3) $\dim T^{\perp}\mathcal{A} = \dim vT^{\perp}\mathcal{A}$ *where* v *is the vertical projector.*
(4) $TTM = T\mathcal{A} \oplus vT^{\perp}\mathcal{A}$.

Indeed, (2) is obtained by taking the orthogonal of (1). On the other hand, (2) means that $v\xi \neq 0$ for any $\xi \in T^{\perp}\mathcal{A}$, $\xi \neq 0$, that is the restriction of v to $T^{\perp}\mathcal{A}$ is injective, which is equivalent to $\dim T^{\perp}\mathcal{A} = \dim vT^{\perp}\mathcal{A}$. Then (1), (2) and (3) are equivalent.

Finally, suppose that (2) holds and let $v\xi \in vT^{\perp}\mathcal{A} \cap T\mathcal{A}$, with $\xi \in T^{\perp}\mathcal{A}$. We have $g_{\Gamma}(v\xi, \xi) = 0$, because $v\xi \in T\mathcal{A}$. Now g_{Γ} is adapted, so $g_{\Gamma}(v\xi, \xi) = g_{\Gamma}(v\xi, v\xi) = 0$. Since g_{Γ} is positive definite, that implies $v\xi = 0$, hence $(vT^{\perp}\mathcal{A}) \cap T\mathcal{A} = 0$. But $\dim T^{\perp}\mathcal{A} = \dim vT^{\perp}\mathcal{A}$, hence $TTM = T\mathcal{A} \oplus vT^{\perp}\mathcal{A}$ which shows that (4). holds. The converse is evident.

Corollary 3.4 (cf. [13]). *Let E be a convex Lagrangian, S the canonical spray and $\mathcal{A}$ a non-holonomic admissible constraint. Then there exists a unique vector field $\xi \in T^{\perp}\mathcal{A}$ such that the vector field defined on $\mathcal{A}$ by $S' = S - v\xi$ is an admissible spray.*

Indeed, we have just to decompose S following the direct sum $TTM = T\mathcal{A} \oplus vT^{\perp}\mathcal{A}$: $S = S' + v\xi$. If ξ' is another vector field in $T^{\perp}\mathcal{A}$ such that $S = S' + v\xi'$, one has $v(\xi - \xi') = 0$, i.e. $\xi - \xi' \in T^h \cap T^{\perp}\mathcal{A} = \{0\}$, hence ξ is unique. On the other hand, S' is an admissible spray, because $JS' = JS = C$.

4. Geometry of the constraints

Let E be a convex Lagrangian and Γ an arbitrary connection (for example, the Lagrangian connection associated with the canonical spray as in Section 2). In what follows, we shall denote by τ and ν the projectors on $T\mathcal{A}$ and $vT^{\perp}\mathcal{A}$ defined by the direct sum

$$TTM = T\mathcal{A} \oplus vT^{\perp}\mathcal{A}$$

associated with Γ (cf. Proposition 3.3). [8] Note that, since ν is vertical-valued, we have

$$J\nu = 0 \quad \text{and} \quad J\tau = J.$$

The vertical endomorphism J induces a (1–1) tensor field J' on $\mathcal{A}$ (cf. [10]) defined by

$$J' = \tau J \mid_{T\mathcal{A}}.$$

Proposition 4.1.
(1) $(J')^2 = 0$
(2) $\operatorname{Im} J' = T^v \cap T\mathcal{A} \subset \operatorname{Ker} J'$, $\operatorname{Ker} J' = (T^v + FT^{\perp}\mathcal{A}) \cap T\mathcal{A}$.

[8] Note that τ and ν depend on E and on the choise of Γ and, as [10] remarks, the objects which we shall construct are rather *mechanical* quantities than *geometrical*.

Indeed,

$$JJ' = J\tau J = J^2 = 0 \quad \text{and} \quad J'J = \tau J\tau J = \tau J^2 = 0$$

hence

$$\operatorname{Im} J' \subset \operatorname{Ker} J\,|_{T\mathcal{A}} = \operatorname{Im} J\,|_{T\mathcal{A}} \subset \operatorname{Ker} J'$$

and so $(J')^2 = 0$.

If $X \in \operatorname{Im} J'$ then $X \in T\mathcal{A}$ and $X \in \operatorname{Ker} J$. Therefore $X \in T^v \cap T\mathcal{A}$. Conversely, if $X \in T^v \cap T\mathcal{A}$, then $X = \tau X$ and $X = JY$, whence

$$X = \tau JY = \tau J\tau Y = J'\tau Y;$$

so $X \in \operatorname{Im} J'$. This proves the first part of (2).

Let $X \in \operatorname{Ker} J'$, i.e. $\tau JX = 0$. This implies that $JX \in vT^\perp\mathcal{A}$, hence there exists $\xi \in T^\perp\mathcal{A}$ such that $JX = v\xi$. Let us set $X = FY$. We have $JX = vY$, whence $vY = v\xi$. From this it follows that $Y - \xi \in T^h$ and so $Y \in T^\perp\mathcal{A} + T^h$. Therefore

$$X \in F(T^\perp\mathcal{A} + T^h) = FT^\perp\mathcal{A} + T^v.$$

Conversely, if $X \in (T^v + FT^\perp\mathcal{A}) \cap T\mathcal{A}$, then $X = JY + F\xi$ (with $\xi \in T^\perp\mathcal{A}$). Now $X = \tau X$, hence

$$X = \tau JY + \tau F\xi = J'Y + \tau F\xi;$$

therefore

$$JX = J\tau F\xi = JF\xi = v\xi$$

and then $\tau JX = \tau v\xi = 0$, i.e. $J'X = 0$. This proves the second part of (2).

Definition 4.2. The induced connection on the constraint $\mathcal{A}$ is the (1–1) tensor field on $\mathcal{A}$ defined by:

$$\Gamma' = \tau\Gamma\,|_{T\mathcal{A}}\,.$$

We set

$$h' = \tfrac{1}{2}(I + \Gamma') \quad \text{and} \quad v' = \tfrac{1}{2}(I - \Gamma').$$

Proposition 4.3.
1. $\Gamma'^2 = I$, $h'^2 = h'$, $v'^2 = v'$,
2. $J'\Gamma' = J'$, $\Gamma'J' = -J'$,
3. $T\mathcal{A} = T^{h'} \oplus T^{v'}$, *where* $T^{v'} := \operatorname{Ker} h' = \operatorname{Im} v' = T^v \cap T\mathcal{A}$ *and* $T^{h'} := \operatorname{Im} h' = \operatorname{Ker} v' = (T^h \oplus T^\perp\mathcal{A}) \cap T\mathcal{A}$.

Proof. Since $\operatorname{Im} \nu \subset T^v$, we get $\Gamma\nu = -\nu$, and then $\tau\Gamma\nu = 0$, i.e.

$$\tau\Gamma\tau = \tau\Gamma.$$

Therefore

1. $\Gamma'^2 = \tau\Gamma\tau\Gamma\ |_{T\mathcal{A}} = \tau\Gamma^2\ |_{T\mathcal{A}} = \tau\ |_{T\mathcal{A}} = I_{T\mathcal{A}}$. It follows that h' and v' are projectors.
2. $J'\Gamma' = \tau J\tau\Gamma\ |_{T\mathcal{A}} = \tau J\Gamma\ |_{T\mathcal{A}} = \tau J\ |_{T\mathcal{A}} = J'$. In the same way one proves that $\Gamma' J' = -J'$.
3. Since h' and v' are projectors, $T\mathcal{A} = T^{h'} \oplus T^{v'}$. On the other hand, let $X \in T\mathcal{A}$ be such that $h'X = 0$, i.e. $\tau hX = 0$. This implies $X \in vT^{\perp}\mathcal{A}$, hence $X \in T^v \cap T\mathcal{A}$.

Conversely, if $X \in T^v \cap T\mathcal{A}$, $hX = 0$, then $\tau' X = \tau hX = 0$.

Finally, $\operatorname{Im} h' = \operatorname{Ker} v'$. Now $v'X = 0$ means that $\tau vX = 0$, i.e. $vX = v\xi$ (with $\xi \in T^{\perp}\mathcal{A}$), which is equivalent to $X - \xi \in T^h$. Then $X \in T^h \oplus T^{\perp}\mathcal{A}$ and therefore $\operatorname{Im} h' = (T^h \oplus T^{\perp}\mathcal{A}) \cap T\mathcal{A}$. □

Remark. *Since $\mathcal{A}$ is fibered on M by the restriction to $\mathcal{A}$ of the projection of TM on M, and $T^v \cap T\mathcal{A}$ is the vertical bundle with respect to this fibration, Property 3, means that Γ' is a connection (which in general is not linear) on the fiber bundle $\mathcal{A} \to M$. That accounts for the terminology.The definitions of geodesics and parallel fields (cf. Section 1) adjust naturally to this case. An $\mathcal{A}$-valued vector field z along a curve $\gamma : I \to M$ is parallel if its velocity $z' = \gamma_* \circ d/dt$ is horizontal, i.e. if $v'z'(t) = 0$. An admissible geodesic is a curve on M such that its velocity is $\mathcal{A}$-valued and parallel with respect to Γ'.*

Definition 4.4. Let $(M, E, \mathcal{A})$ be a constrained Lagrangian system with convex Lagrangian E. The canonical connection adapted to the constraint $\mathcal{A}$ is the connection induced by the Lagrangian connection constructed in Section 2.

Theorem 4.5. *Let $(M, E, \mathcal{A})$ be a Lagrangian constrained system, with convex Lagrangian E (eventually non-homogeneous) and an ideal non-holonomic constraint $\mathcal{A}$. Let Γ' be the canonical connection adapted to the constraint. Then*

1. *The solutions of the d'Euler–Lagrange equations are the geodesics of Γ' with velocity in $\mathcal{A}$.*
2. *One has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) = 0$$

for all curves $\gamma : I \to M$ and all parallel vector fields $z(t) \in \mathcal{A}$ along γ. In other words, the Hamiltonian is preserved by parallel transport.

Proof. We have just to prove that $(h'X) \cdot \mathcal{H} = 0$ for any $X \in T\mathcal{A}$, i.e. $\Omega(S, h'X) = 0$ for any $X \in T\mathcal{A}$. Now

$$\Omega(S, h'X) = g_\Gamma(FS, h'X) = -g_\Gamma(C, h'X).$$

Indeed,

$$FC = FJS = hS = S$$

because S is the spray of the connection Γ and so $hS = S$. Therefore $FS = -C$. On the other hand, $\operatorname{Im} h' = (T^h + T^{\perp}\mathcal{A}) \cap T\mathcal{A}$, hence

$$h'X = hZ + \xi, \quad \text{with } \xi \in T^{\perp}\mathcal{A}.$$

Then

$$\Omega(S, h'X) = -g_\Gamma(C, hZ) - g_\Gamma(C, \xi).$$

But since g_Γ is adapted to the connection Γ, we have $g_\Gamma(C, hX) = 0$ because C is vertical and hX is horizontal. But $g_\Gamma(C, \xi) = 0$ because $C \in T\mathcal{A}$ and $\xi \in T^{\perp}\mathcal{A}$. Then $\Omega(S, h'X) = 0$. □

Example. *Let us consider a regular Lagrangian $E = E_2 + U$, where E_2 is quadratic and $U = U(x)$ is a function on M. E_2 determines a Riemannian metric g on M (g is the polarization of E_2 : $E_2(z) = (1/2)g(z, z)$). The Hamiltonian is $\mathcal{H} = E_2 - U$ and $\Omega = \mathrm{d}\mathrm{d}_J E = \mathrm{d}\mathrm{d}_J E_2$, because $\mathrm{d}_J U = 0$. On the other hand, $g(C, C) = \mathcal{L}_C\mathcal{H} = 2E_2$.*

Let S be the canonical spray defined by $i_S\Omega = -\mathrm{d}\mathcal{H}$. Put

$$S = S_2 + V,$$

where S_2 is the canonical spray of the Levi-Civita connection defined by $i_{S_2}\mathrm{d}\mathrm{d}_J E_2 = -\mathrm{d}E_2$. Now

$$i_V \mathrm{d}\mathrm{d}_J E_2 = \mathrm{d}U$$

and locally

$$V = V^\alpha \frac{\partial}{\partial y^\alpha} \quad \text{with } V^\alpha = g^{\alpha\beta}\frac{\partial U}{\partial x^\beta}.$$

Since the components of V do not depend on the variables y, we get

$$[J, S] = [J, S_2],$$

i.e. the natural connection is the Levi-Civita connection $\Gamma_2 = [J, S_2]$.

Therefore the canonical connection is

$$\Gamma = \Gamma_2 + T,$$

where T the strong torsion given by the formula (5) of Section 2.

We have $[C, S] = [C, S_2] + [C, V] = S_2 - V$, hence

$$S^* = -2V \qquad i_{S^*}\Omega = -2\mathrm{d}U$$

and

$$g(S^*, C) = -2g(V, C) = -2i_V\Omega(S) = -2\mathcal{L}_S U.$$

Then

$$T = \frac{2}{g(C,C)^2}[g(C,C)\mathrm{d}U \otimes C + g(C,C)i_C\Omega \otimes V - (\mathcal{L}_S U)i_C\Omega \otimes C]$$

i.e. locally,

$$T^i_j = \frac{2}{(g_{\alpha\beta}y^\alpha y^\beta)^2}g^{ik}[(g_{\lambda\mu}y^\lambda y^\mu)((\partial_k U)y_j + (\partial_j U)y_k) - (y^\alpha \partial_\alpha U)y_j y_k],$$

where $y_i = g_{ij}y^j$. Then the coefficients of the canonical connection are:

$$\Gamma^i_j = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} y^k - \frac{1}{2}T^i_j,$$

where the $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ are the coefficients of the Levi-Civita connection.

Let us consider now an admissible constraint $\mathcal{A}$ of codimension 1, in TM, defined by the equation

$$f(x, y) = 0.$$

The admissibility condition means that in this case there exists a vertical vector JZ which is not tangent to $\mathcal{A}$, i.e. such that $JZ \cdot f \neq 0$. Then $\mathcal{A}$ is admissible if and only if

$$\mathrm{d}_J f \neq 0$$

at any point of $\mathcal{A}$. A normal vector field ξ verifies $g_\Gamma(\xi, X) = 0$ for any $X \in T\mathcal{A}$, i.e. for X such that $\mathrm{d}f(X) = 0$. Then $g_\Gamma(\xi, \cdot)$ is proportional to $\mathrm{d}f$, or, in other words: $\Omega(\xi, FY) = \mathrm{d}f(Y)$, i.e.

$$i_\xi \Omega = -\mathrm{d}_F f.$$

Then we have

$$i_{v\xi}\Omega = i_\xi i_v \Omega - i_v i_\xi \Omega = i_\xi \Omega + \mathrm{d}_{Fv} f = -\mathrm{d}_F f + \mathrm{d}_F v f = -\mathrm{d}_{Fh} f = \mathrm{d}_J f$$

(because $i_v\Omega = \Omega$, Γ being Lagrangian). Therefore the vector field $v\xi$ such that

$$i_{v\xi}\Omega = \mathrm{d}_J f$$

splits $vT^\perp\mathcal{A}$. Locally

$$v\xi = g^{\alpha\beta}\frac{\partial f}{\partial y^\beta}\frac{\partial}{\partial y^\alpha}.$$

Now $\mathcal{A}$ is admissible, hence: $\mathrm{d}_J f \neq 0$, i.e. $v\xi \neq 0$. We have $g(v\xi, v\xi) = \Omega(v\xi, F\xi) = \mathrm{d}_J f(F\xi) = v\xi \cdot f$. Then

$$\| v\xi \|^2 = g^{\alpha\beta}\frac{\partial f}{\partial y^\alpha}\frac{\partial f}{\partial y^\beta}.$$

Let $X = X_{\mathcal{A}} + \lambda v\xi$ be the splitting of X with respect to the direct sum $TTM = T\mathcal{A} \oplus vT^{\perp}\mathcal{A}$ where $X_{\mathcal{A}}$ is the tangent part to $\mathcal{A}$. Imposing that $X_{\mathcal{A}} \in T\mathcal{A}$, i.e. $X_{\mathcal{A}} \cdot f = 0$, we find $\lambda = (\mathrm{d}f)(X)/\| v\xi \|^2$. The projectors ν and τ are

$$\nu = \frac{\mathrm{d}f}{\| v\xi \|^2} \otimes v\xi \quad \text{and} \quad \tau = I - \frac{\mathrm{d}f}{\| v\xi \|^2} \otimes v\xi.$$

A straightforward computation gives the connection Γ' induced on the constraint.

References

[1] S. Benenti, Geometrical aspects of dynamics of non-holonomic systems, Conference Journées Rélativistes 1987, Chambery.

[2] A. Frölicher, A. Nijenhuis, Theory of vector-valued differential forms, Proc. Kon. Ned. Akad. A 59 (1956) 338–359.

[3] F. Gallissot, Les formes extérieures en mécanique, Thèse, Durand, Chartres, 1954.

[4] J. Klein, Espaces variationnels et mécanique, Thèse, Durand, Chartres, 1961.

[5] C. Godbillon, Géométrie différentielle et mécanique, Hermann, Paris, 1969.

[6] J. Grifone, Structure presque-tangente et connexions I, II, Ann. Inst. Fourier (1972), XXII (1), pp. 287–334; XXII (3), pp. 291–388.

[7] A.D. Lewis, R.M. Murray, Variational principles for constrained systems: theory and experiment, Internet J. Non-linear Mech. 30 (6) (1995) pp. 793, 815.

[8] C.M. Marle, Réduction of constrained mechanical systems and stability of relative equilibria, Comm. Math. Phys. 174 (1995) 295–318.

[9] E. Massa, E. Pagani, Classical dynamics of non-holonomic systems: a geometrical approach, Ann. Inst. H. Poincaré 55 (1) (1991) 511–544.

[10] E. Massa, E. Pagani, A new look at classical mechanics of constrained systems, Ann. Inst. H. Poincaré 66 (1) (1997) 1–36.

[11] M. de Léon, D.M. de Diego, On the Geometry of non-holonomic Lagrangian systems, J. Math. Phys. 37 (7) (1996) 3389–3414.

[12] D.J. Saunders, W. Sarlet, F. Cantrijn, A geometrical framework for the study of non-holonomic Lagrangian systems, I, II, J. Phys. A 28 (1995) 3253–3268; 29 (1996) 4265–4274.

[13] A.M. Vershik, L.D. Faddeev, Differential Geometry and Lagrangian mechanics with constraints, Soviet Phys. Dokl. 17 (1972) 34–36.